MAT 402 S26
PROBLEM SET 1

I. Let W be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and let $T : W \to W$ be a linear transformation such that $||Tx|| = ||x||$ for all $x \in W$. Show that T is a bijection and that

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for all } x, y \in W. \tag{1}$$

(If T satisfies (1) then we call T an *orthogonal transformation*, and we write $T \in O(W, \langle \cdot, \cdot \rangle)$.)

II. Let $\Gamma := (A, V, t, \langle \cdot, \cdot \rangle)$ be a Galilean structure. Show that if $F : A \to A$ is a Galilean transformation of Γ to itself then F is a bijection that satisfies

$$F(\lambda a_1 + (1 - \lambda) a_2) = \lambda F(a_1) + (1 - \lambda) F(a_2) \quad \text{for all } a_1, a_2 \in A \text{ and } \lambda \in [0, 1]. \tag{2}$$

(A map satisfying (2) is called an *affine transformation*.)

III. Let S be a real vector space of dimension 3 with an inner product $\langle \cdot, \cdot \rangle$. Consider the Galilean structure

$$(A, V, t, \langle \cdot, \cdot \rangle)$$

where $V = S \times \mathbb{R}$, $t : V \to \mathbb{R}$ is the Cartesian projection to the second factor and $A = V$ as a set, with the action of V on A being the canonical action of a vector space on itself:

$$A \times V \ni (x, v) \mapsto x + v \in A.$$

For $v \in S$, $r \in V$ and $T \in O(S, \langle \cdot, \cdot \rangle)$, define the maps $\beta_v, \delta_r, \rho_T : A \to A$ by

$$\beta_v(x, s) = (x + sv, s), \quad \delta_r(x, s) = (x, s) + r \quad \text{and} \quad \rho_T(x, s) = (Tx, s).$$

a. Show that these maps are Galilean transformations.

b. Show if $F : A \to A$ is a Galilean transformation then there exist unique $v \in S$, $r \in V$ and $T \in O(S, \langle \cdot, \cdot \rangle)$ such that

$$F = \beta_V \circ \delta_r \circ \rho_T.$$

IV. Recall that a *motion* in a Galilean spacetime $(A, V, t, \langle \cdot, \cdot \rangle)$ is a map $x : I \to A$, and that a motion is called a *world line* if $t \circ x : I \to \mathbb{R}$ is an increasing function. Show that the following are equivalent.

a. The motion $x : I \to A$ is a world line.

b. There exists a Galilean coordinate system $\Phi : A \to \mathbb{R}^3 \times \mathbb{R}$ such that $\Phi \circ x$ is a world line.

c. For every Galilean coordinate system $\Phi : A \to \mathbb{R}^3 \times \mathbb{R}$ the motion $\Phi \circ x$ is a world line.

d. There exists a Galilean coordinate system $\Phi : A \to \mathbb{R}^3 \times \mathbb{R}$, an increasing surjective function $\sigma : \Sigma \to I$ for some interval $\Sigma \subset \mathbb{R}$ and a map $p : \Sigma \to \mathbb{R}^3$ such that

$$\Phi \circ x \circ \sigma(s) = (p(s), s)$$

for all $s \in \Sigma \subset \mathbb{R}$.

V. Prove that the following claims are consequences of the principles of Newtonian Determinacy and Galilean Relativity, as we have formulated them mathematically.

a. Show that a mechanical system consisting of a single point has no acceleration in any inertial coordinate system. (This statement is sometimes called *Netwon's First Law*, or *The Law of Intertia*.)

b. Show that if a mechanical system consisting of two points whose velocities at some initial time and in some inertial coordinate system are 0 is allowed to evolve then, in this same coordinate system, the line connecting the two points at any other time is independent of that time, i.e., the moving points will remain on the line that connected the two initial positions.

c. Consider a mechanical system consisting of three points. Fix an inertial coordinate system, thereby determining three functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}^3$, and let $P \subset \mathbb{R}^3$ be the affine plane containing the three points $x_1(0), x_2(0), x_3(0)$. Suppose that the initial velocities $\dot{x}_1(0), \dot{x}_2(0), \dot{x}_3(0)$ are parallel to the affine plane P. Show that the affine plane $P(t) \subset \mathbb{R}^3$ containing the three points $x_1(t), x_2(t), x_3(t)$ is independent of t.